Math 211 - Bonus Exercise 5 (please discuss on Forum)

1) Prove that if $\{G_1, \dots, G_n\}$ and $\{G'_1, \dots, G'_n\}$ denote the same collection of abelian groups, but perhaps in a different order, then there exists an isomorphism

$$G_1 \times \cdots \times G_n \cong G'_1 \times \cdots \times G'_n$$

Use this to construct an action of the symmetric group S_n on $G \times \cdots \times G$ (the n-fold direct product of an abelian group G) by automorphisms.

2) We have seen that any short exact sequence of abelian groups $0 \to K \to G \to \mathbb{Z}^r \to 0$ splits. But is it also true that any short exact sequence of abelian groups $0 \to \mathbb{Z}^r \to G \to L \to 0$ also splits?

3) Any homomorphism

$$f : \mathbb{Z}^r \to \mathbb{Z}^r$$

can be completely determined by a $r \times r$ matrix A with integer coefficients, by

$$f\left(\begin{pmatrix} k_1 \\ \vdots \\ k_r \end{pmatrix}\right) = A \begin{pmatrix} k_1 \\ \vdots \\ k_r \end{pmatrix}$$

(we write tuples $(k_1, \dots, k_r)$ in column form). Which property purely in terms of A tells you if f is injective? What about the index of Im f in $\mathbb{Z}^r$, can that be expressed purely in terms of A?

4) Prove that if G is a finitely generated abelian group, then it is not divisible: this means that there exists some $g \in G$ and some $n > 0$ such that $\frac{g}{n}$ does not exist in G (i.e. $g \neq nh, \forall h \in G$).